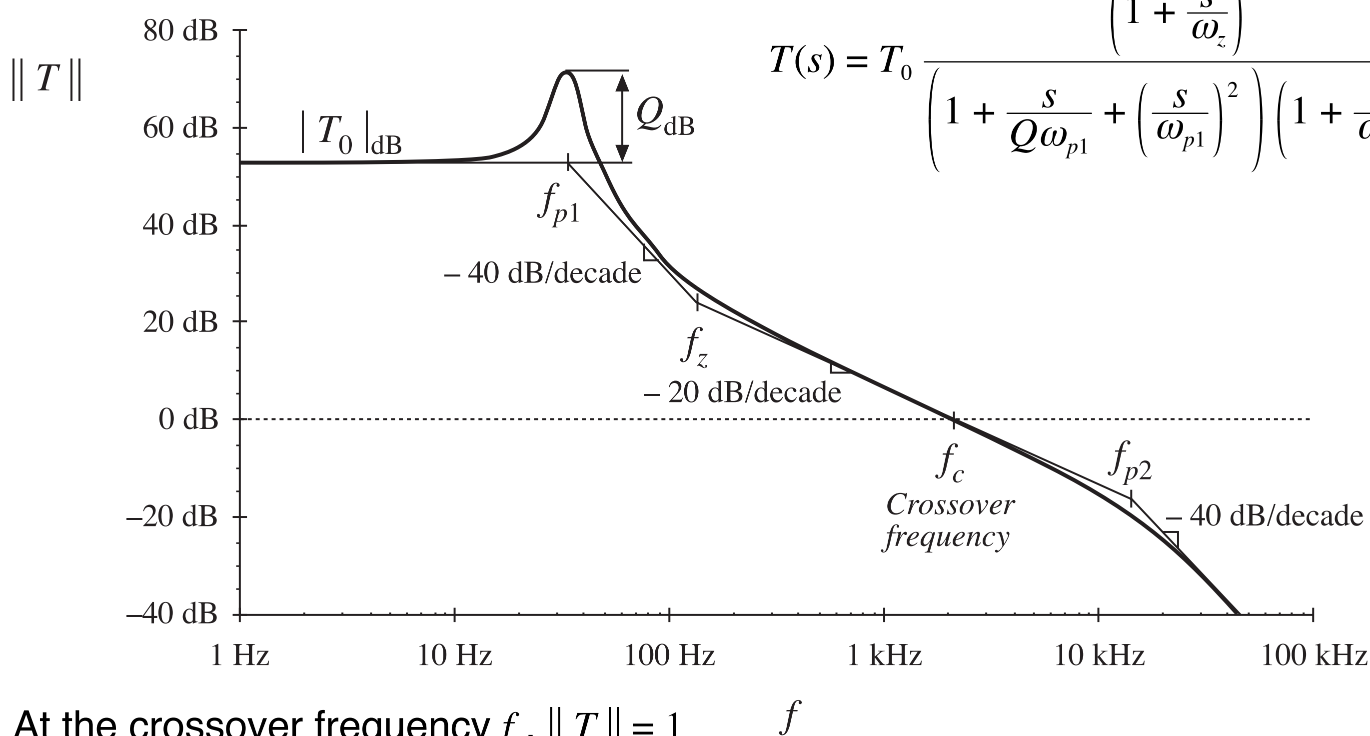
9.3. Construction of the important quantities $1/(1+T)$ and $T/(1+T)$

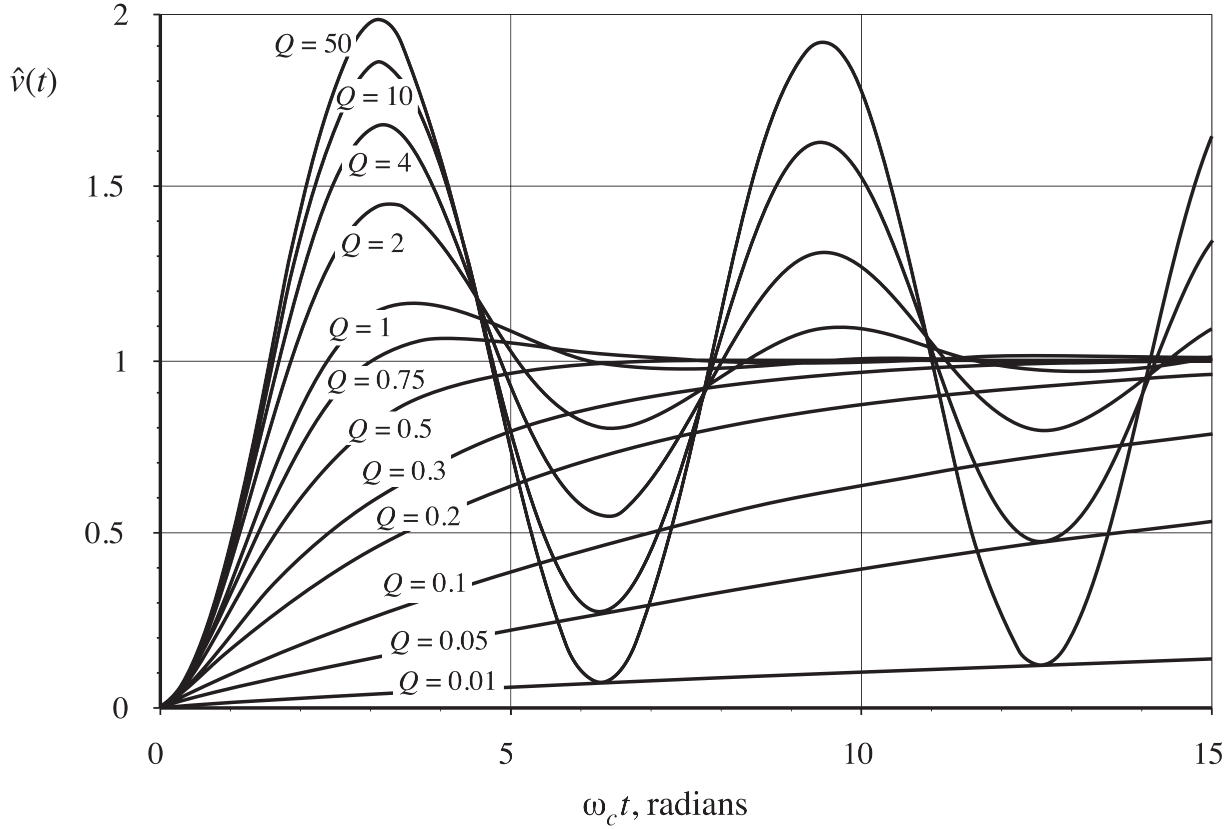
Example

$$T(s) = T_0 \frac{\left(1 + \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{Q\omega_{p1}} + \left(\frac{s}{\omega_{p1}}\right)^2\right)\left(1 + \frac{s}{\omega_{p2}}\right)}$$



At the crossover frequency f_c, $\| T \| = 1$

Transient response vs. damping factor



9.4. Stability

Even though the original open-loop system is stable, the closed-loop transfer functions can be unstable and contain right half-plane poles. Even when the closed-loop system is stable, the transient response can exhibit undesirable ringing and overshoot, due to the high Q -factor of the closed-loop poles in the vicinity of the crossover frequency.

When feedback destabilizes the system, the denominator $(1+T(s))$ terms in the closed-loop transfer functions contain roots in the right half-plane (i.e., with positive real parts). If $T(s)$ is a rational fraction of the form $N(s) / D(s)$, where $N(s)$ and $D(s)$ are polynomials, then we can write

$$\frac{T(s)}{1+T(s)} = \frac{\frac{N(s)}{D(s)}}{1+\frac{N(s)}{D(s)}} = \frac{N(s)}{N(s)+D(s)}$$

$$\frac{1}{1+T(s)} = \frac{1}{1+\frac{N(s)}{D(s)}} = \frac{D(s)}{N(s)+D(s)}$$

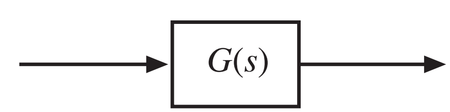
- Could evaluate stability by evaluating $N(s) + D(s)$, then factoring to evaluate roots. This is a lot of work, and is not very illuminating.

Effect of feedback on transfer function poles

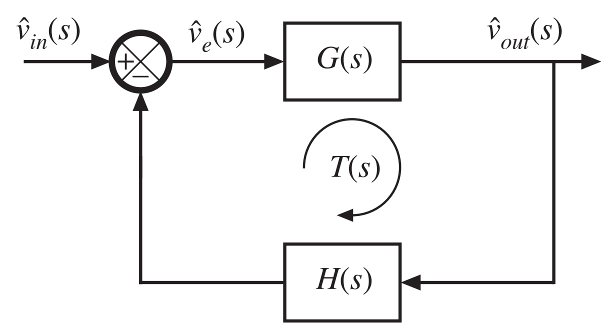
Feedback moves the poles of the system transfer functions

- Good news: we can use feedback to alter the poles and improve the frequency response
- Bad news: if you're not careful, feedback can move the poles into the right half of the complex *s*-plane (poles have positive real parts), leading to an unstable system

Open loop



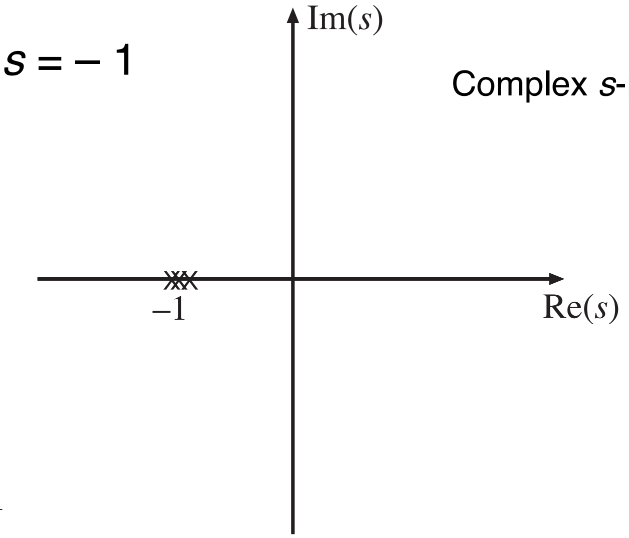
Closed loop



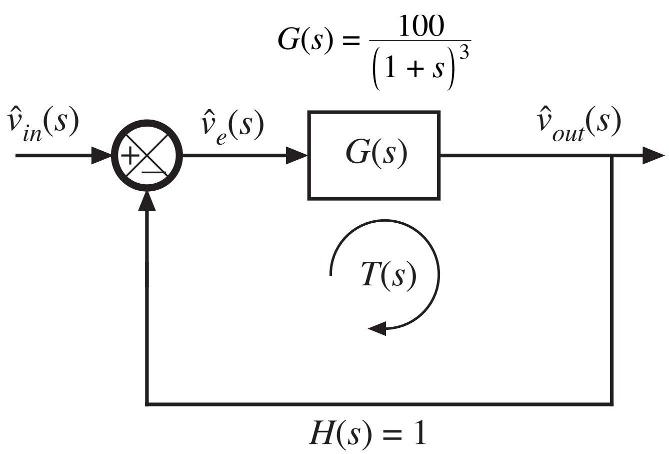
Example

The gain $G(s)$ below has three poles at $s = -1$

$$G(s) = \frac{100}{\left(1 + s\right)^3}$$



Add a simple feedback loop:



How does the feedback change the poles?

Exact closed-loop transfer function

For our simple example, the closed-loop transfer function is

$$\frac{\hat{v}_{out}}{\hat{v}_{in}} = \frac{1}{H}\frac{T}{1+T} = \frac{G}{1+G} = \frac{\dfrac{100}{\left(1+s\right)^3}}{1+\dfrac{100}{\left(1+s\right)^3}} = \frac{100}{101 + 3s + 3s^2 + s^3}$$

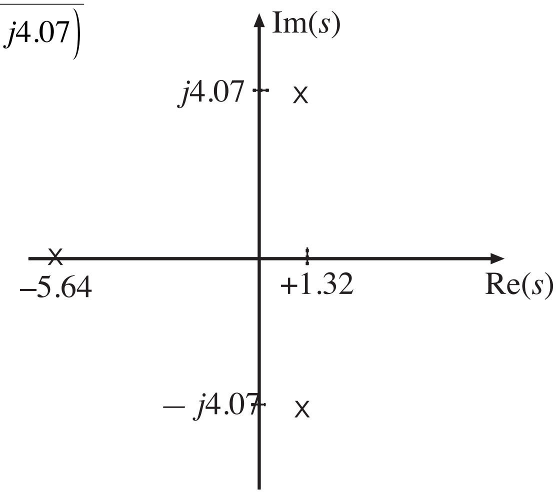
Factor denominator numerically:

$$\frac{\hat{v}_{out}}{\hat{v}_{in}} = \frac{100}{101 + 3s + 3s^2 + s^3} = \frac{100}{\left(s+5.64\right)\left(s-1.32-j4.07\right)\left(s-1.32+j4.07\right)}$$

which has poles at s = – 5.64 (LHP)

and at s = +1.32 ± j4.07 (RHP)

The RHP poles indicate that the closed-loop system is unstable.



Transient response of closed-loop system

One can take the inverse Laplace Transform to find the output waveform $\hat{v}_{out}(t)$ for a given input. The resulting expression has terms that depend on the poles, of the form

$$\hat{v}_{out}(t) = K_1 e^{-5.64t} + K_2 e^{(1.32 - j4.07)t} + K_2^* e^{(1.32 + j4.07)t}$$

The terms with positive real exponents, corresponding to the RHP poles, lead to growing oscillations that are unstable responses.

Reason: *the inverse Laplace transform of* $K_2 e^{(1.32 - j4.07)t} + K_2^* e^{(1.32 + j4.07)t}$ *is*

$$\left\| K_2 \right\| e^{1.32t} \cos\left(4.07t + \angle K_2\right)$$

Determination of stability directly from $T(s)$

- Nyquist stability theorem: general result.
- A special case of the Nyquist stability theorem: the phase margin test

  Allows determination of closed-loop stability (i.e., whether $1/(1+T(s))$ contains RHP poles) directly from the magnitude and phase of $T(s)$.

  A good design tool: yields insight into how $T(s)$ should be shaped, to obtain good performance in transfer functions containing $1/(1+T(s))$ terms.

9.4.1. The phase margin test

A test on $T(s)$, to determine whether $1/(1+T(s))$ contains RHP poles.

The crossover frequency f_c is defined as the frequency where

$$\| T(j2\pi f_c) \| = 1 \Rightarrow 0\text{dB}$$

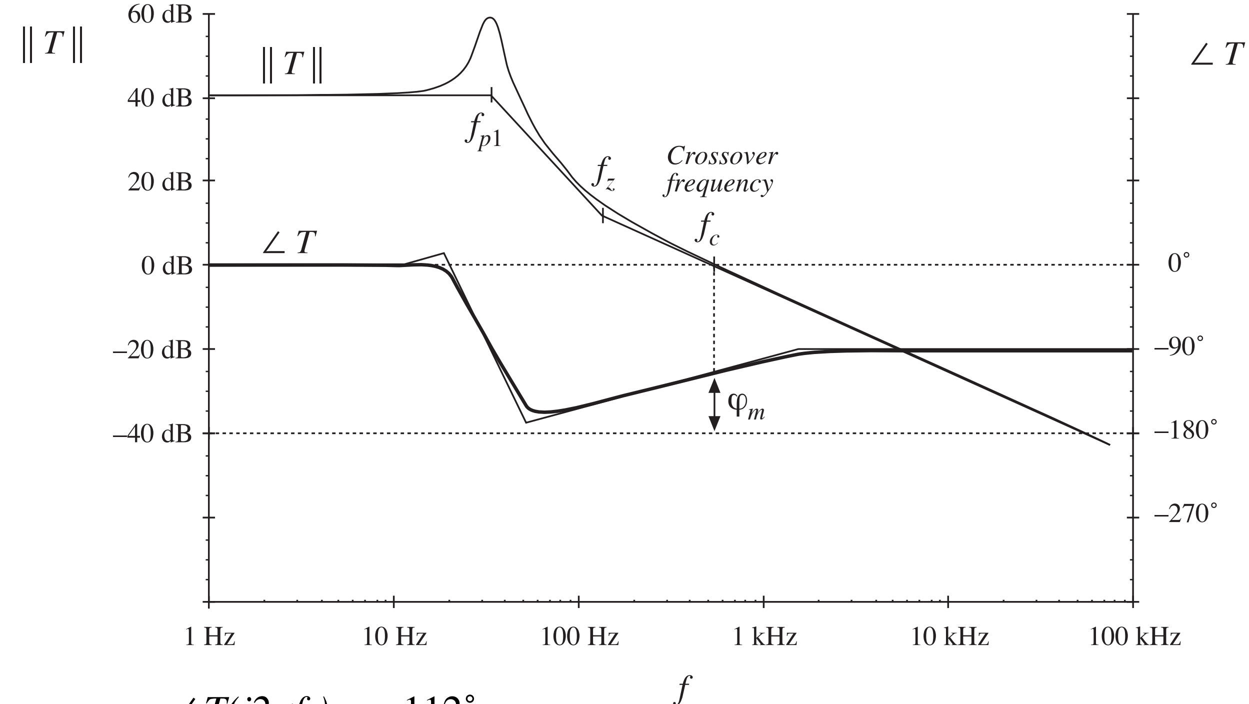
The phase margin φ_m is determined from the phase of $T(s)$ at f_c , as follows:

$$\varphi_m = 180^\circ + \angle T(j2\pi f_c)$$

If there is exactly one crossover frequency, and if $T(s)$ contains no RHP poles, then

the quantities $T(s)/(1+T(s))$ and $1/(1+T(s))$ contain no RHP poles whenever the phase margin φ_m is positive.

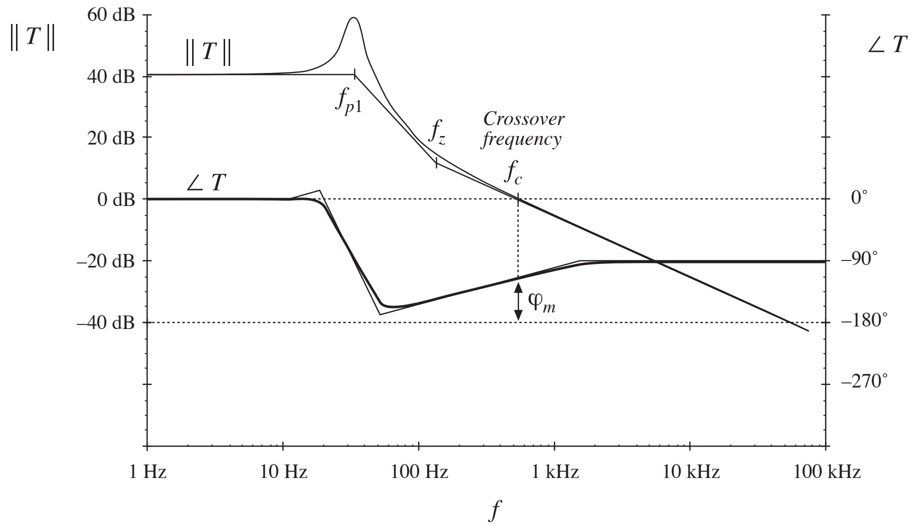
Example: a loop gain leading to a stable closed-loop system



$\angle T(j2\pi f_c) = -112^\circ$

$\varphi_m = 180^\circ - 112^\circ = +68^\circ$

Computation of crossover frequency



1. The expression for $T(s)$ is:

$$T(s) = T_0 \frac{\left(1 + \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{Q\omega_{p1}} + \left(\frac{s}{\omega_{p1}}\right)^2\right)}$$

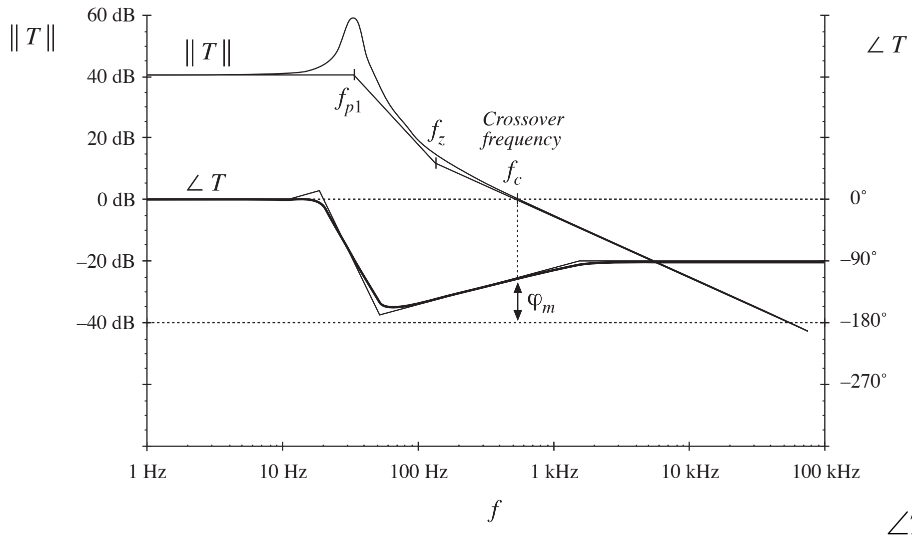
2. Write the equation of the asymptote for $f > f_z$:

$$\|T(j\omega)\| = T_0 \frac{\left(\frac{\omega}{\omega_z}\right)}{\left(\frac{\omega}{\omega_{p1}}\right)^2} = T_0 \frac{f_{p1}^2}{f_z f}$$

3. Equate to 1, and solve for f (= f_c):

$$f_c = T_0 \frac{f_{p1}^2}{f_z}$$

Computation of phase



$$T(s) = T_0 \frac{\left(1 + \frac{s}{\omega_z}\right)}{\left(1 + \frac{s}{Q\omega_{p1}} + \left(\frac{s}{\omega_{p1}}\right)^2\right)}$$

Exact expression for phase:

$$\angle T(j\omega) = \tan^{-1}\left(\frac{\omega}{\omega_z}\right) - \tan^{-1}\left[\frac{\frac{1}{Q}\left(\frac{\omega}{\omega_{p1}}\right)}{1 - \left(\frac{\omega}{\omega_{p1}}\right)^2}\right]$$

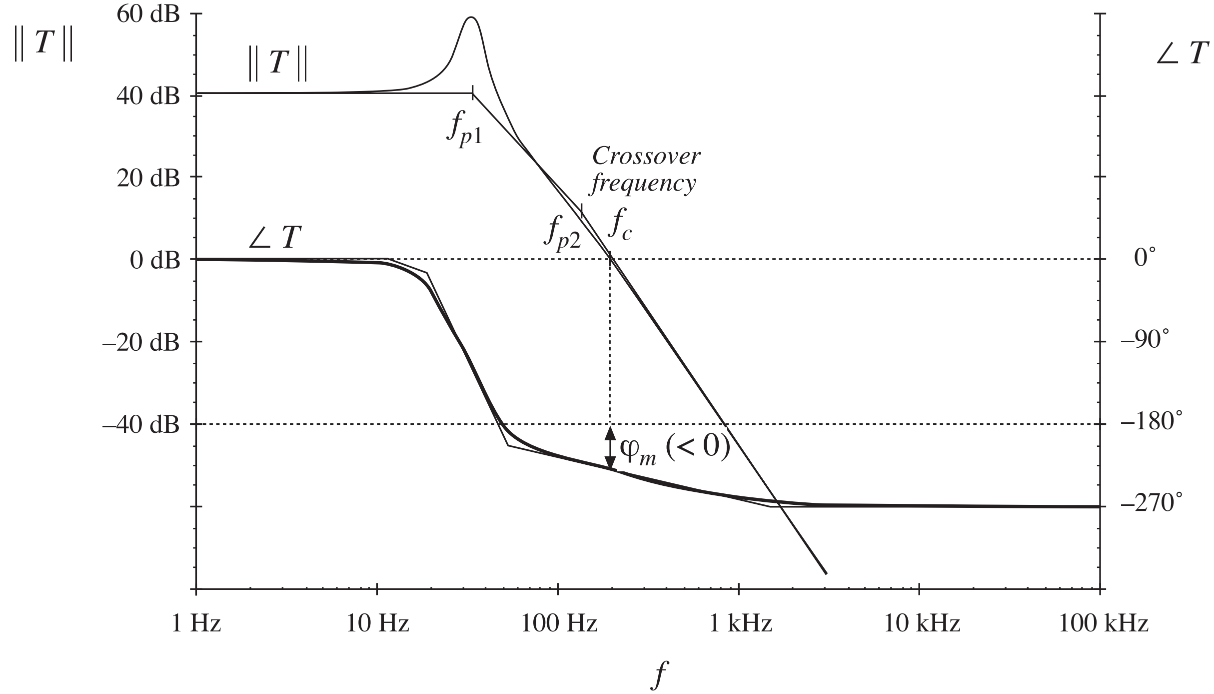
Expression for phase asymptote over frequency range as illustrated near f_c:

$$\angle T(j\omega) \approx (+45^\circ/\mathrm{dec}) \log_{10}\left(\frac{\omega}{\omega_z/10}\right) - 180^\circ$$

Evaluate one of the above to find $\angle T(j\omega_c)$, then compute phase margin:

$$\varphi_m = 180^\circ + \angle T(j\omega_c)$$

Example: a loop gain leading to an unstable closed-loop system



$\angle T(j2\pi f_c) = -230^\circ$

$\varphi_m = 180^\circ - 230^\circ = -50^\circ$