7.3 State Space Averaging

- A formal method for deriving the small-signal ac equations of a switching converter
- Equivalent to the modeling method of the previous sections
- Uses the state-space matrix description of linear circuits
- Often cited in the literature
- A general approach: if the state equations of the converter can be written for each subinterval, then the small-signal averaged model can always be derived
- Computer programs exist which utilize the state-space averaging method

7.3.1 The state equations of a network

- A canonical form for writing the differential equations of a system
- If the system is linear, then the derivatives of the *state variables* are expressed as linear combinations of the system independent inputs and state variables themselves
- The physical state variables of a system are usually associated with the storage of energy
- For a typical converter circuit, the physical state variables are the inductor currents and capacitor voltages
- Other typical physical state variables: position and velocity of a motor shaft
- At a given point in time, the values of the state variables depend on the previous history of the system, rather than the present values of the system inputs
- To solve the differential equations of a system, the initial values of the state variables must be specified

State equations of a linear system, in matrix form

A canonical matrix form:

$$\mathbf{K}\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{E}\,\mathbf{u}(t)$$

State vector $\mathbf{x}(t)$ contains inductor currents, capacitor voltages, etc.:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix}, \qquad \frac{d\mathbf{x}(t)}{dt} = \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \vdots \end{bmatrix}$$

Input vector $\mathbf{u}(t)$ contains independent sources such as $v_g(t)$

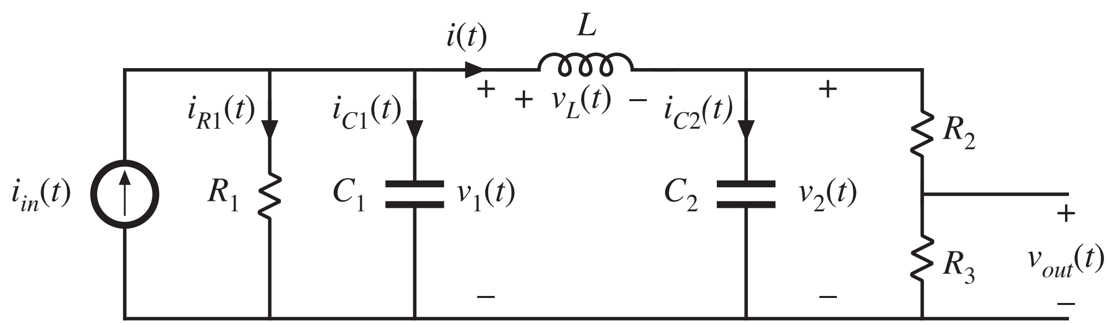
Output vector $\mathbf{y}(t)$ contains other dependent quantities to be computed, such as $i_g(t)$

Matrix $\mathbf{K}$ contains values of capacitance, inductance, and mutual inductance, so that $\mathbf{K}\, d\mathbf{x}/dt$ is a vector containing capacitor currents and inductor winding voltages. These quantities are expressed as linear combinations of the independent inputs and state variables. The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{E}$ contain the constants of proportionality.

Example

State vector

$$\mathbf{x}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ i(t) \end{bmatrix}$$



Matrix **K**

$$\mathbf{K} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix}$$

Input vector

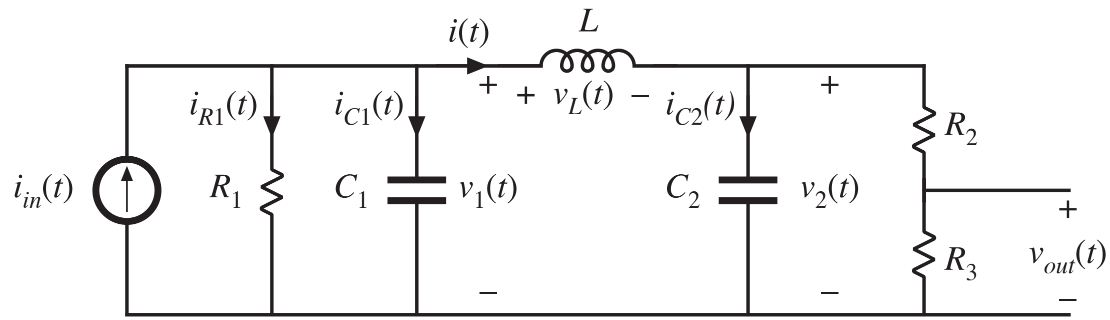
$$\mathbf{u}(t) = \begin{bmatrix} i_{in}(t) \end{bmatrix}$$

Choose output vector as

$$\mathbf{y}(t) = \begin{bmatrix} v_{out}(t) \\ i_{R1}(t) \end{bmatrix}$$

To write the state equations of this circuit, we must express the inductor voltages and capacitor currents as linear combinations of the elements of the $\mathbf{x}(t)$ and $\mathbf{u}(t)$ vectors.

Circuit equations



Find i_{C1} via node equation: $$i_{C1}(t) = C_1 \frac{dv_1(t)}{dt} = i_{in}(t) - \frac{v_1(t)}{R} - i(t)$$

Find i_{C2} via node equation: $$i_{C2}(t) = C_2 \frac{dv_2(t)}{dt} = i(t) - \frac{v_2(t)}{R_2 + R_3}$$

Find v_L via loop equation: $$v_L(t) = L \frac{di(t)}{dt} = v_1(t) - v_2(t)$$

Equations in matrix form

The same equations:

$$i_{C1}(t) = C_1 \frac{dv_1(t)}{dt} = i_{in}(t) - \frac{v_1(t)}{R} - i(t)$$

$$i_{C2}(t) = C_2 \frac{dv_2(t)}{dt} = i(t) - \frac{v_2(t)}{R_2 + R_3}$$

$$v_L(t) = L \frac{di(t)}{dt} = v_1(t) - v_2(t)$$

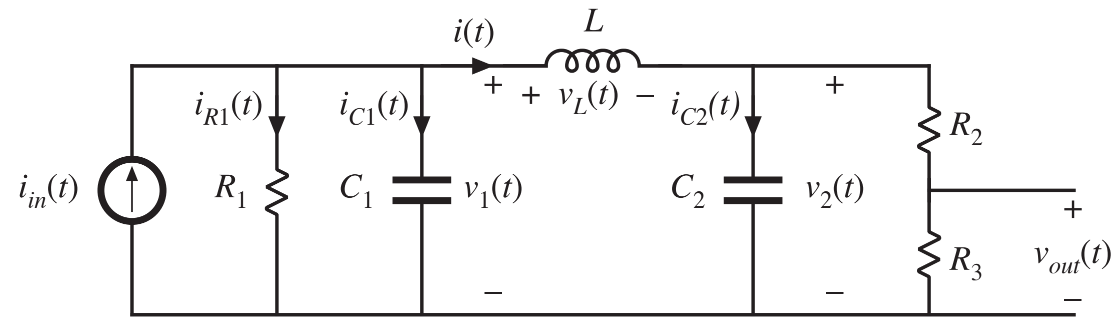
Express in matrix form:

$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} \frac{dv_1(t)}{dt} \\ \frac{dv_2(t)}{dt} \\ \frac{di(t)}{dt} \end{bmatrix}}_{\frac{d\mathbf{x}(t)}{dt}} = \underbrace{\begin{bmatrix} -\frac{1}{R_1} & 0 & -1 \\ 0 & -\frac{1}{R_2 + R_3} & 1 \\ 1 & -1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} v_1(t) \\ v_2(t) \\ i(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} i_{in}(t) \end{bmatrix}}_{\mathbf{u}(t)}$$

$$\mathbf{K} \frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$

Output (dependent signal) equations

$$\mathbf{y}(t) = \begin{bmatrix} v_{out}(t) \\ i_{R1}(t) \end{bmatrix}$$



Express elements of the vector $\mathbf{y}$ as linear combinations of elements of $\mathbf{x}$ and $\mathbf{u}$:

$$v_{out}(t) = v_2(t)\,\frac{R_3}{R_2 + R_3}$$

$$i_{R1}(t) = \frac{v_1(t)}{R_1}$$

Express in matrix form

The same equations:

$$v_{out}(t) = v_2(t)\,\frac{R_3}{R_2 + R_3}$$

$$i_{R1}(t) = \frac{v_1(t)}{R_1}$$

Express in matrix form:

$$\underbrace{\begin{bmatrix} v_{out}(t) \\ i_{R1}(t) \end{bmatrix}}_{\mathbf{y}(t)} = \underbrace{\begin{bmatrix} 0 & \dfrac{R_3}{R_2 + R_3} & 0 \\ \dfrac{1}{R_1} & 0 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} v_1(t) \\ v_2(t) \\ i(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} i_{in}(t) \end{bmatrix}}_{\mathbf{u}(t)}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) + \mathbf{E}\,\mathbf{u}(t)$$

7.3.2 The basic state-space averaged model

Given: a PWM converter, operating in continuous conduction mode, with two subintervals during each switching period.

During subinterval 1*,* when the switches are in position 1, the converter reduces to a linear circuit that can be described by the following state equations:

$$\mathbf{K}\,\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_1\,\mathbf{x}(t) + \mathbf{B}_1\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_1\,\mathbf{x}(t) + \mathbf{E}_1\,\mathbf{u}(t)$$

During subinterval 2*,* when the switches are in position 2, the converter reduces to another linear circuit, that can be described by the following state equations:

$$\mathbf{K}\,\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_2\,\mathbf{x}(t) + \mathbf{B}_2\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}_2\,\mathbf{x}(t) + \mathbf{E}_2\,\mathbf{u}(t)$$

Equilibrium (dc) state-space averaged model

Provided that the natural frequencies of the converter, as well as the frequencies of variations of the converter inputs, are much slower than the switching frequency, then the state-space averaged model that describes the converter in equilibrium is

$$\mathbf{0} = \mathbf{A}\,\mathbf{X} + \mathbf{B}\,\mathbf{U}$$
$$\mathbf{Y} = \mathbf{C}\,\mathbf{X} + \mathbf{E}\,\mathbf{U}$$

where the averaged matrices are

$$\mathbf{A} = D\,\mathbf{A}_1 + D'\,\mathbf{A}_2$$
$$\mathbf{B} = D\,\mathbf{B}_1 + D'\,\mathbf{B}_2$$
$$\mathbf{C} = D\,\mathbf{C}_1 + D'\,\mathbf{C}_2$$
$$\mathbf{E} = D\,\mathbf{E}_1 + D'\,\mathbf{E}_2$$

and the equilibrium dc components are

$\mathbf{X}$ = *equilibrium (dc) state vector*
$\mathbf{U}$ = *equilibrium (dc) input vector*
$\mathbf{Y}$ = *equilibrium (dc) output vector*
D = *equilibrium (dc) duty cycle*

Solution of equilibrium averaged model

Equilibrium state-space averaged model:

$$\mathbf{0} = \mathbf{A}\,\mathbf{X} + \mathbf{B}\,\mathbf{U}$$
$$\mathbf{Y} = \mathbf{C}\,\mathbf{X} + \mathbf{E}\,\mathbf{U}$$

Solution for $\mathbf{X}$ and $\mathbf{Y}$:

$$\mathbf{X} = -\,\mathbf{A}^{-1}\,\mathbf{B}\,\mathbf{U}$$
$$\mathbf{Y} = \left(-\,\mathbf{C}\,\mathbf{A}^{-1}\,\mathbf{B} + \mathbf{E}\right)\mathbf{U}$$

Small-signal ac state-space averaged model

$$\mathbf{K}\,\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{A}\,\hat{\mathbf{x}}(t) + \mathbf{B}\,\hat{\mathbf{u}}(t) + \left\{ \left(\mathbf{A}_1 - \mathbf{A}_2\right)\mathbf{X} + \left(\mathbf{B}_1 - \mathbf{B}_2\right)\mathbf{U} \right\} \hat{d}(t)$$

$$\hat{\mathbf{y}}(t) = \mathbf{C}\,\hat{\mathbf{x}}(t) + \mathbf{E}\,\hat{\mathbf{u}}(t) + \left\{ \left(\mathbf{C}_1 - \mathbf{C}_2\right)\mathbf{X} + \left(\mathbf{E}_1 - \mathbf{E}_2\right)\mathbf{U} \right\} \hat{d}(t)$$

where

$$\hat{\mathbf{x}}(t) = small - signal\ (ac)\ perturbation\ in\ state\ vector$$

$$\hat{\mathbf{u}}(t) = small - signal\ (ac)\ perturbation\ in\ input\ vector$$

$$\hat{\mathbf{y}}(t) = small - signal\ (ac)\ perturbation\ in\ output\ vector$$

$$\hat{d}(t) = small - signal\ (ac)\ perturbation\ in\ duty\ cycle$$

So if we can write the converter state equations during subintervals 1 and 2, then we can always find the averaged dc and small-signal ac models

7.3.3 Discussion of the state-space averaging result

As in Sections 7.1 and 7.2, the low-frequency components of the inductor currents and capacitor voltages are modeled by averaging over an interval of length T_s. Hence, we define the average of the state vector as:

$$\left\langle \mathbf{x}(t) \right\rangle_{T_s} = \frac{1}{T_s} \int_t^{t+T_s} \mathbf{x}(\tau)\, d\tau$$

The low-frequency components of the input and output vectors are modeled in a similar manner.

By averaging the inductor voltages and capacitor currents, one obtains:

$$\mathbf{K} \frac{d\left\langle \mathbf{x}(t) \right\rangle_{T_s}}{dt} = \left(d(t)\, \mathbf{A}_1 + d'(t)\, \mathbf{A}_2 \right) \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \left(d(t)\, \mathbf{B}_1 + d'(t)\, \mathbf{B}_2 \right) \left\langle \mathbf{u}(t) \right\rangle_{T_s}$$

Change in state vector during first subinterval

During subinterval 1, we have

$$\mathbf{K}\,\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_1\,\mathbf{x}(t) + \mathbf{B}_1\,\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}_1\,\mathbf{x}(t) + \mathbf{E}_1\,\mathbf{u}(t)$$

So the elements of $\mathbf{x}(t)$ change with the slope

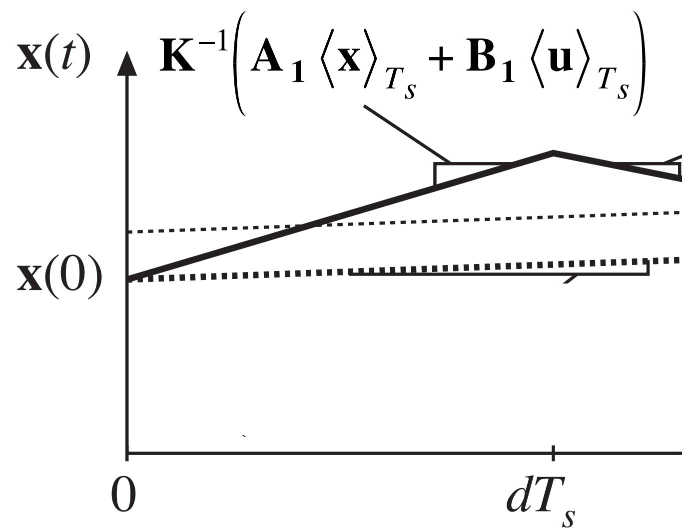
$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{K}^{-1}\left(\mathbf{A}_1\,\mathbf{x}(t) + \mathbf{B}_1\,\mathbf{u}(t)\right)$$

Small ripple assumption: the elements of $\mathbf{x}(t)$ and $\mathbf{u}(t)$ do not change significantly during the subinterval. Hence the slopes are essentially constant and are equal to

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{K}^{-1}\left(\mathbf{A}_1\left\langle\mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B}_1\left\langle\mathbf{u}(t)\right\rangle_{T_s}\right)$$

Change in state vector during first subinterval

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{K}^{-1}\left(\mathbf{A_1}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B_1}\left\langle \mathbf{u}(t)\right\rangle_{T_s}\right)$$



Net change in state vector over first subinterval:

$$\underbrace{\mathbf{x}(dT_s)}_{\textit{final value}} = \underbrace{\mathbf{x}(0)}_{\textit{initial value}} + \underbrace{\left(dT_s\right)}_{\textit{interval length}} \underbrace{\mathbf{K}^{-1}\left(\mathbf{A_1}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B_1}\left\langle \mathbf{u}(t)\right\rangle_{T_s}\right)}_{\textit{slope}}$$

Change in state vector during second subinterval

Use similar arguments.

State vector now changes with the essentially constant slope

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{K}^{-1}\left(\mathbf{A}_2 \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \mathbf{B}_2 \left\langle \mathbf{u}(t) \right\rangle_{T_s}\right)$$

The value of the state vector at the end of the second subinterval is therefore

$$\underbrace{\mathbf{x}(T_s)}_{\substack{\textit{final}\\ \textit{value}}} = \underbrace{\mathbf{x}(dT_s)}_{\substack{\textit{initial}\\ \textit{value}}} + \underbrace{\left(d'T_s\right)}_{\substack{\textit{interval}\\ \textit{length}}} \underbrace{\mathbf{K}^{-1}\left(\mathbf{A}_2 \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \mathbf{B}_2 \left\langle \mathbf{u}(t) \right\rangle_{T_s}\right)}_{\textit{slope}}$$

Net change in state vector over one switching period

We have:

$$\mathbf{x}(dT_s) = \mathbf{x}(0) + \left(dT_s\right)\mathbf{K}^{-1}\left(\mathbf{A_1}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B_1}\left\langle \mathbf{u}(t)\right\rangle_{T_s}\right)$$

$$\mathbf{x}(T_s) = \mathbf{x}(dT_s) + \left(d'T_s\right)\mathbf{K}^{-1}\left(\mathbf{A_2}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B_2}\left\langle \mathbf{u}(t)\right\rangle_{T_s}\right)$$

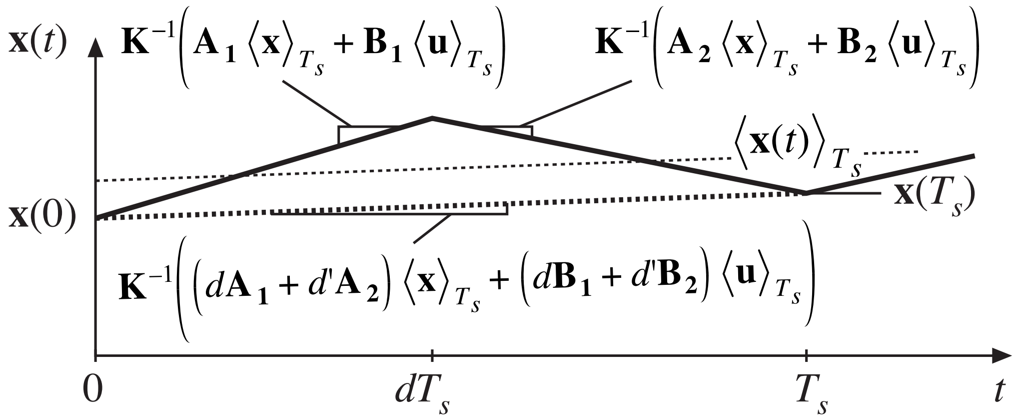
Eliminate $\mathbf{x}(dT_s)$, to express $\mathbf{x}(T_s)$ directly in terms of $\mathbf{x}(0)$:

$$\mathbf{x}(T_s) = \mathbf{x}(0) + dT_s\mathbf{K}^{-1}\left(\mathbf{A_1}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B_1}\left\langle \mathbf{u}(t)\right\rangle_{T_s}\right) + d'T_s\mathbf{K}^{-1}\left(\mathbf{A_2}\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \mathbf{B_2}\left\langle \mathbf{u}(t)\right\rangle_{T_s}\right)$$

Collect terms:

$$\mathbf{x}(T_s) = \mathbf{x}(0) + T_s\mathbf{K}^{-1}\left(d(t)\mathbf{A_1} + d'(t)\mathbf{A_2}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + T_s\mathbf{K}^{-1}\left(d(t)\mathbf{B_1} + d'(t)\mathbf{B_2}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}$$

Approximate derivative of state vector



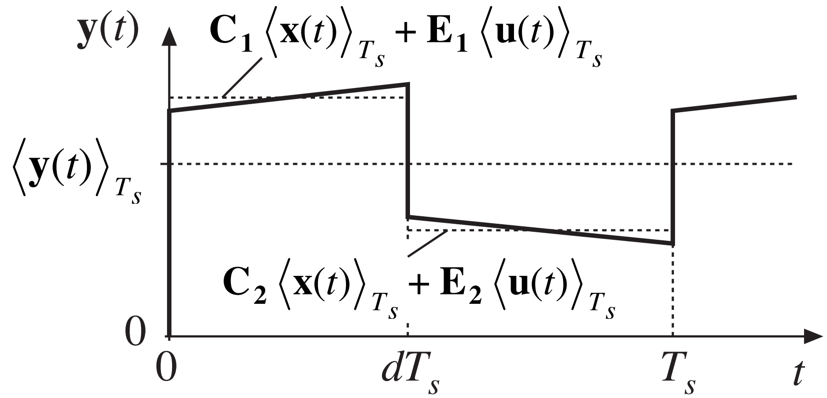
Use Euler approximation:

$$\frac{d\left\langle\mathbf{x}(t)\right\rangle_{T_s}}{dt}\approx\frac{\mathbf{x}(T_s)-\mathbf{x}(0)}{T_s}$$

We obtain:

$$\mathbf{K}\frac{d\left\langle\mathbf{x}(t)\right\rangle_{T_s}}{dt}=\left(d(t)\,\mathbf{A_1}+d'(t)\,\mathbf{A_2}\right)\left\langle\mathbf{x}(t)\right\rangle_{T_s}+\left(d(t)\,\mathbf{B_1}+d'(t)\,\mathbf{B_2}\right)\left\langle\mathbf{u}(t)\right\rangle_{T_s}$$

Low-frequency components of output vector



Remove switching harmonics by averaging over one switching period:

$$\left\langle \mathbf{y}(t) \right\rangle_{T_s} = d(t) \left(\mathbf{C_1} \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \mathbf{E_1} \left\langle \mathbf{u}(t) \right\rangle_{T_s} \right) + d'(t) \left(\mathbf{C_2} \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \mathbf{E_2} \left\langle \mathbf{u}(t) \right\rangle_{T_s} \right)$$

Collect terms:

$$\left\langle \mathbf{y}(t) \right\rangle_{T_s} = \left(d(t)\, \mathbf{C_1} + d'(t)\, \mathbf{C_2} \right) \left\langle \mathbf{x}(t) \right\rangle_{T_s} + \left(d(t)\, \mathbf{E_1} + d'(t)\, \mathbf{E_2} \right) \left\langle \mathbf{u}(t) \right\rangle_{T_s}$$

Averaged state equations: quiescent operating point

The averaged (nonlinear) state equations:

$$\mathbf{K}\frac{d\left\langle \mathbf{x}(t)\right\rangle_{T_s}}{dt} = \left(d(t)\,\mathbf{A_1} + d'(t)\,\mathbf{A_2}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \left(d(t)\,\mathbf{B_1} + d'(t)\,\mathbf{B_2}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}$$

$$\left\langle \mathbf{y}(t)\right\rangle_{T_s} = \left(d(t)\,\mathbf{C_1} + d'(t)\,\mathbf{C_2}\right)\left\langle \mathbf{x}(t)\right\rangle_{T_s} + \left(d(t)\,\mathbf{E_1} + d'(t)\,\mathbf{E_2}\right)\left\langle \mathbf{u}(t)\right\rangle_{T_s}$$

The converter operates in equilibrium when the derivatives of all elements of $\langle \mathbf{x}(t) \rangle_{T_s}$ are zero. Hence, the converter quiescent operating point is the solution of

$$\mathbf{0} = \mathbf{A}\,\mathbf{X} + \mathbf{B}\,\mathbf{U}$$

$$\mathbf{Y} = \mathbf{C}\,\mathbf{X} + \mathbf{E}\,\mathbf{U}$$

where

$$\mathbf{A} = D\,\mathbf{A}_1 + D'\,\mathbf{A}_2$$
$$\mathbf{B} = D\,\mathbf{B_1} + D'\,\mathbf{B_2}$$
$$\mathbf{C} = D\,\mathbf{C_1} + D'\,\mathbf{C_2}$$
$$\mathbf{E} = D\,\mathbf{E_1} + D'\,\mathbf{E_2}$$

and

$\mathbf{X}$ = *equilibrium (dc) state vector*
$\mathbf{U}$ = *equilibrium (dc) input vector*
$\mathbf{Y}$ = *equilibrium (dc) output vector*
D = *equilibrium (dc) duty cycle*

Averaged state equations: perturbation and linearization

Let $\left\langle \mathbf{x}(t) \right\rangle_{T_s} = \mathbf{X} + \hat{\mathbf{x}}(t)$ with $\left\| \mathbf{U} \right\| >> \left\| \hat{\mathbf{u}}(t) \right\|$

$\left\langle \mathbf{u}(t) \right\rangle_{T_s} = \mathbf{U} + \hat{\mathbf{u}}(t)$ $\qquad D >> \left| \hat{d}(t) \right|$

$\left\langle \mathbf{y}(t) \right\rangle_{T_s} = \mathbf{Y} + \hat{\mathbf{y}}(t)$ $\qquad \left\| \mathbf{X} \right\| >> \left\| \hat{\mathbf{x}}(t) \right\|$

$d(t) = D + \hat{d}(t) \quad \Rightarrow \quad d'(t) = D' - \hat{d}(t)$ $\qquad \left\| \mathbf{Y} \right\| >> \left\| \hat{\mathbf{y}}(t) \right\|$

Substitute into averaged state equations:

$$\mathbf{K} \frac{d\left(\mathbf{X}+\hat{\mathbf{x}}(t)\right)}{dt} = \left(\left(D+\hat{d}(t)\right) \mathbf{A}_1 + \left(D'-\hat{d}(t)\right) \mathbf{A}_2\right) \left(\mathbf{X}+\hat{\mathbf{x}}(t)\right) + \left(\left(D+\hat{d}(t)\right) \mathbf{B}_1 + \left(D'-\hat{d}(t)\right) \mathbf{B}_2\right) \left(\mathbf{U}+\hat{\mathbf{u}}(t)\right)$$

$$\left(\mathbf{Y}+\hat{\mathbf{y}}(t)\right) = \left(\left(D+\hat{d}(t)\right) \mathbf{C}_1 + \left(D'-\hat{d}(t)\right) \mathbf{C}_2\right) \left(\mathbf{X}+\hat{\mathbf{x}}(t)\right) + \left(\left(D+\hat{d}(t)\right) \mathbf{E}_1 + \left(D'-\hat{d}(t)\right) \mathbf{E}_2\right) \left(\mathbf{U}+\hat{\mathbf{u}}(t)\right)$$

Averaged state equations: perturbation and linearization

$$\underbrace{\mathbf{K}\frac{d\hat{\mathbf{x}}(t)}{dt}}_{\textit{first–order ac}} = \underbrace{\left(\mathbf{AX}+\mathbf{BU}\right)}_{\textit{dc terms}} + \underbrace{\mathbf{A}\hat{\mathbf{x}}(t)+\mathbf{B}\hat{\mathbf{u}}(t)+\left\{\left(\mathbf{A}_1-\mathbf{A}_2\right)\mathbf{X}+\left(\mathbf{B}_1-\mathbf{B}_2\right)\mathbf{U}\right\}\hat{d}(t)}_{\textit{first–order ac terms}}$$

$$+ \underbrace{\left(\mathbf{A}_1-\mathbf{A}_2\right)\hat{\mathbf{x}}(t)\hat{d}(t)+\left(\mathbf{B}_1-\mathbf{B}_2\right)\hat{\mathbf{u}}(t)\hat{d}(t)}_{\textit{second–order nonlinear terms}}$$

$$\underbrace{\left(\mathbf{Y}+\hat{\mathbf{y}}(t)\right)}_{\textit{dc + 1st order ac}} = \underbrace{\left(\mathbf{CX}+\mathbf{EU}\right)}_{\textit{dc terms}} + \underbrace{\mathbf{C}\hat{\mathbf{x}}(t)+\mathbf{E}\hat{\mathbf{u}}(t)+\left\{\left(\mathbf{C}_1-\mathbf{C}_2\right)\mathbf{X}+\left(\mathbf{E}_1-\mathbf{E}_2\right)\mathbf{U}\right\}\hat{d}(t)}_{\textit{first–order ac terms}}$$

$$+ \underbrace{\left(\mathbf{C}_1-\mathbf{C}_2\right)\hat{\mathbf{x}}(t)\hat{d}(t)+\left(\mathbf{E}_1-\mathbf{E}_2\right)\hat{\mathbf{u}}(t)\hat{d}(t)}_{\textit{second–order nonlinear terms}}$$

Linearized small-signal state equations

Dc terms drop out of equations. Second-order (nonlinear) terms are small when the small-signal assumption is satisfied. We are left with:

$$\mathbf{K}\,\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{A}\,\hat{\mathbf{x}}(t) + \mathbf{B}\,\hat{\mathbf{u}}(t) + \left\{ \left(\mathbf{A}_1 - \mathbf{A}_2\right)\mathbf{X} + \left(\mathbf{B}_1 - \mathbf{B}_2\right)\mathbf{U} \right\} \hat{d}(t)$$

$$\hat{\mathbf{y}}(t) = \mathbf{C}\,\hat{\mathbf{x}}(t) + \mathbf{E}\,\hat{\mathbf{u}}(t) + \left\{ \left(\mathbf{C}_1 - \mathbf{C}_2\right)\mathbf{X} + \left(\mathbf{E}_1 - \mathbf{E}_2\right)\mathbf{U} \right\} \hat{d}(t)$$

This is the desired result.