9.4.2. The relation between phase margin and closed-loop damping factor

How much phase margin is required?

A small positive phase margin leads to a stable closed-loop system having complex poles near the crossover frequency with high Q. The transient response exhibits overshoot and ringing.

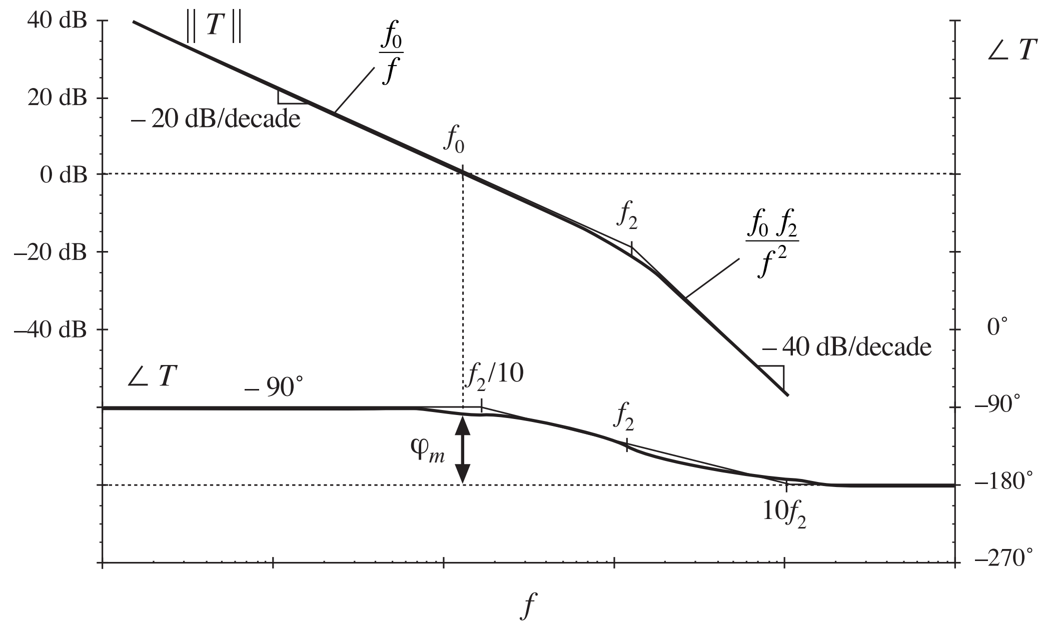
Increasing the phase margin reduces the Q. Obtaining real poles, with no overshoot and ringing, requires a large phase margin.

The relation between phase margin and closed-loop Q is quantified in this section.

A simple second-order system

Consider the case where $T(s)$ can be well-approximated in the vicinity of the crossover frequency as

$$T(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{s}{\omega_2}\right)}$$



Closed-loop response

If

$$T(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

Then

$$\frac{T(s)}{1 + T(s)} = \frac{1}{1 + \frac{1}{T(s)}} = \frac{1}{1 + \frac{s}{\omega_0} + \frac{s^2}{\omega_0 \omega_2}}$$

or,

$$\frac{T(s)}{1 + T(s)} = \frac{1}{1 + \frac{s}{Q\omega_c} + \left(\frac{s}{\omega_c}\right)^2}$$

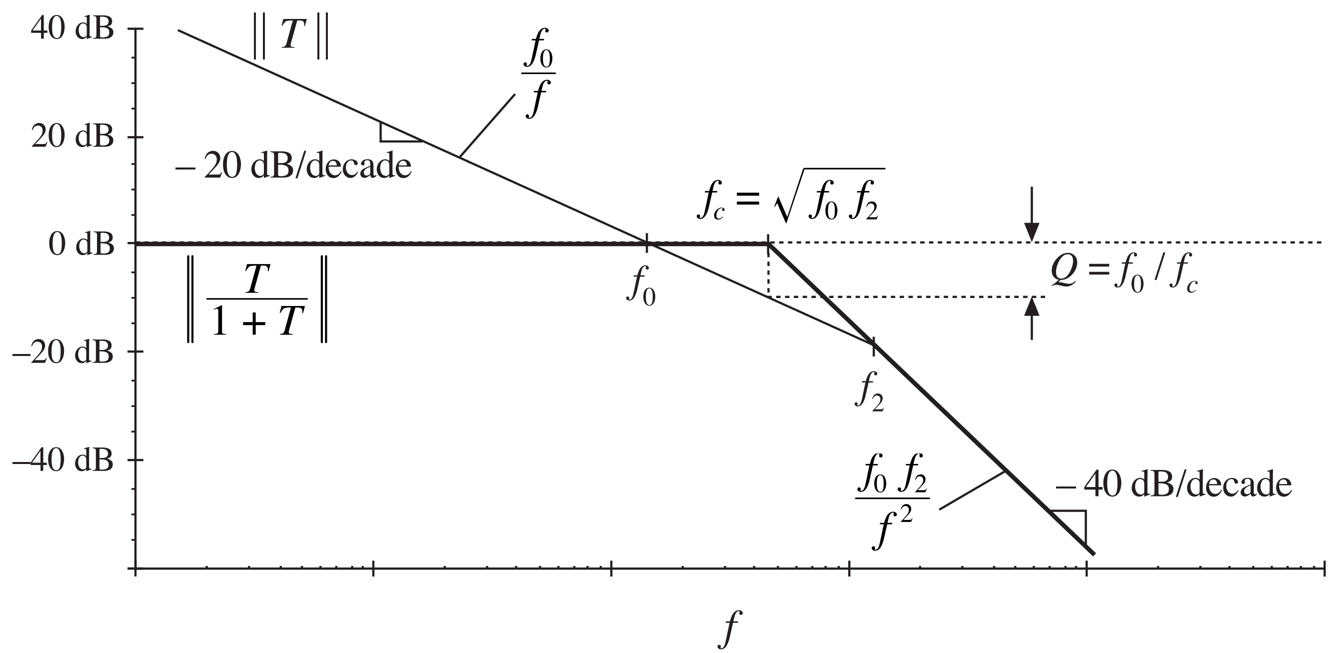
where

$$\omega_c = \sqrt{\omega_0 \omega_2} = 2\pi f_c \qquad Q = \frac{\omega_0}{\omega_c} = \sqrt{\frac{\omega_0}{\omega_2}}$$

Low-Q case

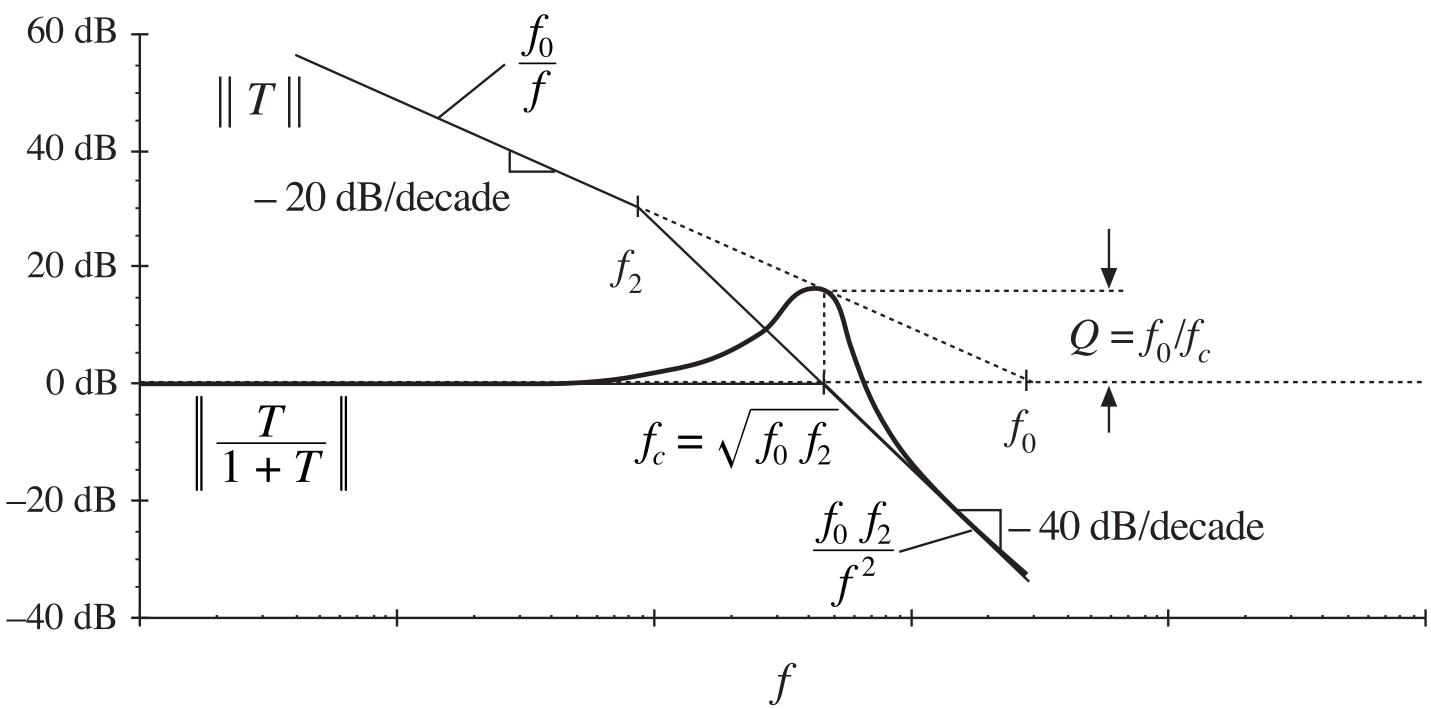
$$Q = \frac{\omega_0}{\omega_c} = \sqrt{\frac{\omega_0}{\omega_2}}$$

low-Q approximation: $Q\,\omega_c = \omega_0$ $\qquad \frac{\omega_c}{Q} = \omega_2$



High-Q case

$$\omega_c = \sqrt{\omega_0 \omega_2} = 2\pi f_c \qquad Q = \frac{\omega_0}{\omega_c} = \sqrt{\frac{\omega_0}{\omega_2}}$$



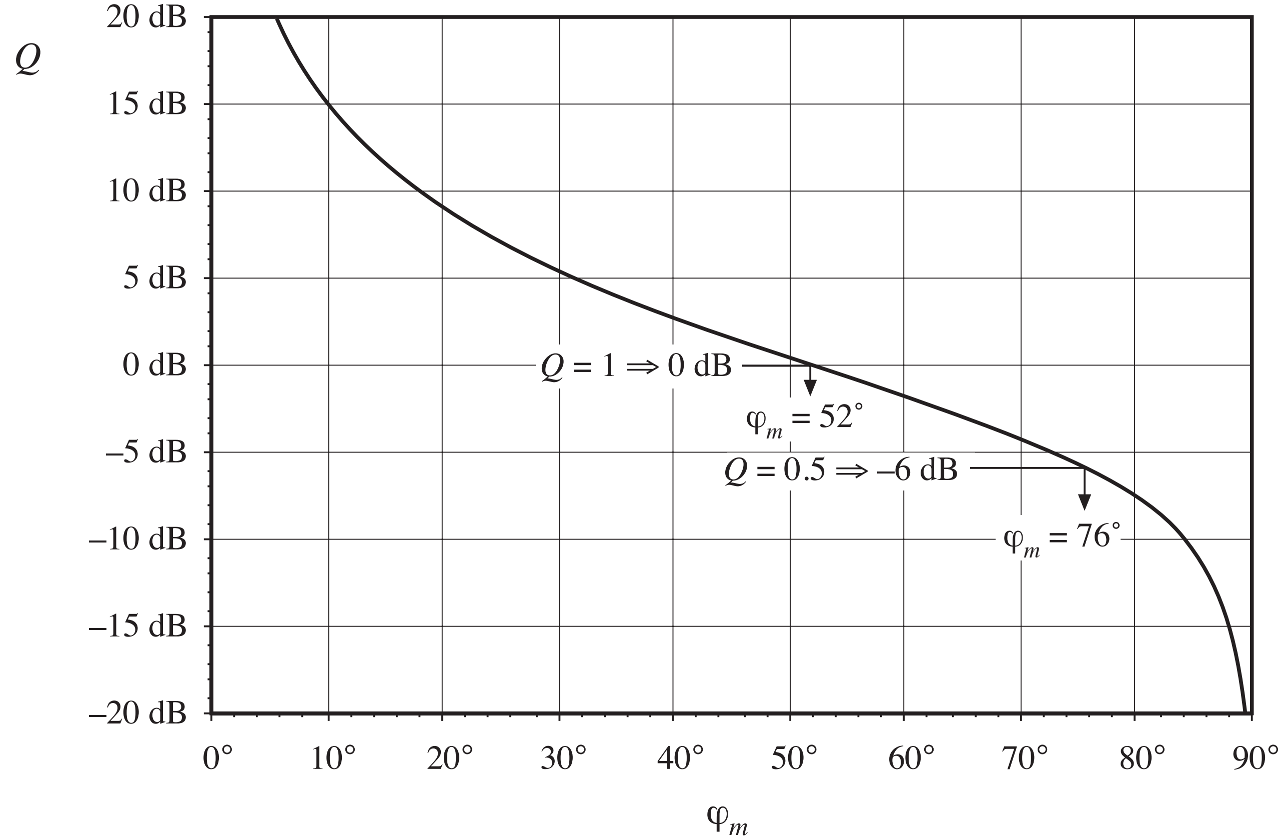
Q vs. φ_m

Solve for exact crossover frequency, evaluate phase margin, express as function of φ_m. Result is:

$$Q = \frac{\sqrt{\cos \varphi_m}}{\sin \varphi_m}$$

$$\varphi_m = \tan^{-1} \sqrt{\frac{1 + \sqrt{1 + 4Q^4}}{2Q^4}}$$

Q vs. φ_m



20 dB
Q
15 dB
10 dB
5 dB
0 dB
–5 dB
–10 dB
–15 dB
–20 dB
Q = 1 ⇒ 0 dB
φm = 52°
Q = 0.5 ⇒ –6 dB
φm = 76°
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
φm

9.4.3. Transient response vs. damping factor

Unit-step response of second-order system $T(s)/(1+T(s))$

$$\hat{v}(t) = 1 + \frac{2Q\, e^{-\omega_c t/2Q}}{\sqrt{4Q^2 - 1}} \sin\left[\frac{\sqrt{4Q^2-1}}{2Q}\,\omega_c\, t + \tan^{-1}\left(\sqrt{4Q^2-1}\right)\right] \qquad Q > 0.5$$

$$\hat{v}(t) = 1 - \frac{\omega_2}{\omega_2 - \omega_1}\, e^{-\omega_1 t} - \frac{\omega_1}{\omega_1 - \omega_2}\, e^{-\omega_2 t} \qquad Q < 0.5$$

$$\omega_1, \omega_2 = \frac{\omega_c}{2Q}\left(1 \pm \sqrt{1 - 4Q^2}\right)$$

For $Q > 0.5$, the peak value is

$$peak\ \hat{v}(t) = 1 + e^{-\pi/\sqrt{4Q^2-1}}$$

Transient response vs. damping factor